Three-fold coverings and hyperelliptic manifolds:
a three-dimensional version of a result of Accola

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We dedicate this paper to the memory of our friend and colleague Marco Reni. We discussed the paper when the first and the third authors were visiting Trieste in February and May 2000, and it was Marco who had a decisive idea for the proof of our main result, shortly before his tragical death in June 2000.

Abstract. It has been proved by Accola that any 3-fold unbranched covering of a Riemann surface of genus two is hyperelliptic (a 2-fold branched covering of the 2-sphere) if the covering is non-regular, and 1-hyperelliptic (a 2-fold branched covering of a torus) if it is regular. In the present paper, we show that the corresponding result holds for closed 3-manifolds when replacing the genus by the Heegaard genus.

1. Introduction

A closed Riemann surface is said to be hyperelliptic if it admits a conformal involution such that the quotient space of the surface by the action of the involution is the 2-sphere $S^2$. Similarly, a closed orientable 3-manifold is hyperelliptic if it admits an involution such that the quotient space of the manifold by the action of the involution is homeomorphic to the 3-sphere $S^3$. It is well known that both Riemann surfaces of genus two and 3-manifolds of Heegaard genus two are hyperelliptic, and it is interesting to ask which properties of hyperelliptic surfaces carry over to hyperelliptic 3-manifolds (see [M],[RZ] for some results on hyperelliptic 3-manifolds, and [FK] for hyperelliptic Riemann surfaces).
It is known that any 2-fold unbranched covering of a Riemann surface of genus two is hyperelliptic ([E],[F],[A1]), and it is proved in [MR] that this result remains true in dimension three: any 2-fold unbranched covering of a closed orientable 3-manifold of (Heegaard) genus two is hyperelliptic. In the present paper, we study the case of 3-fold unbranched coverings. In the case of Riemann surfaces, the following result was obtained in [A, Corollary 1].

**Theorem (Accola)**

Let $S_4 \to S_2$ be an unbranched 3-fold covering of closed Riemann surfaces of genus four and two, respectively.

i) If the covering is regular (Galois), then $S_4$ is 1-hyperelliptic (called elliptic hyperelliptic in [A]: a 2-fold covering of a Riemann surface of genus one, see also [FK, p.249]).

ii) If the covering is non-regular, then $S_4$ is hyperelliptic.

In the present paper we will show that the corresponding result holds also in dimension three. Our main result is as follows.

**Theorem 1**

Let $W_2$ be a 3-manifold of Heegaard genus two, and let $W$ be an unbranched 3-fold covering of $W_2$.

i) If the covering is regular then $W$ is hyperelliptic or a 2-fold branched covering of a 3-manifold of Heegaard genus one (that is of a lens space or of $S^2 \times S^1$).

ii) If the covering is non-regular then $W$ is hyperelliptic.

We recall that a (branched or unbranched) covering is regular if the group of covering transformations acts transitively on each fiber, so the base space is obtained as the quotient by the action of the covering group; equivalently, the covering corresponds to a normal subgroup of the (orbifold) fundamental group of the base space, and the covering group is isomorphic to the factor group.

So there are some common features between hyperelliptic Riemann surfaces and hyperelliptic 3-manifolds when considering the Heegaard genus instead of the genus. However, in general the situation for 3-manifolds is much more complicated. For example, a hyperelliptic Riemann surface has a unique hyperelliptic involution which lies in the center of its automorphism group. On the other hand, hyperelliptic 3-manifolds may have an arbitrarily high number of non-conjugate hyperelliptic involutions (however it has been shown in [RZ] that, in the case of hyperbolic 3-manifolds, there is a universal, in fact quite small bound on the number of conjugacy classes of hyperelliptic involutions, independent of the 3-manifold).
2. Preliminary results

In this section we collect some results which are needed for the proof of Theorem 1.

We first sketch the proof of a Lemma which can be found in standard books on finite transformation groups. To state this Lemma we recall some terminology about abelian groups.

Any finitely generated abelian group $G$ can be expressed as a direct sum $G = \mathbb{Z}^r \oplus T_2 \oplus T_{2'}$, where $T_2$ is the subgroup of elements of $G$ whose order is a power of two, and $T_{2'}$ the subgroup of elements of $G$ whose orders are odd. We call $T_2$ the 2-torsion, $T_{2'}$ the odd torsion and $r$ the rank of $G$. We note that $T_2$ and $T_{2'}$ are characteristic subgroups of $G$ (but not $\mathbb{Z}^r$, in general).

**Lemma**

Let $N$ be a compact 3-manifold and $u$ an involution acting on $N$. Denote by $N_u$ the underlying topological space of the quotient $N/u$ and by $H_1(N)^u$ the subgroup of the elements of the first homology group $H_1(N)$ of $N$ which are fixed by $u$ for the induced action on $H_1(N)$. Then the rank and the odd-torsion of the first homology group $H_1(N_u)$ of $N_u$ are equal, respectively, to the rank and the odd torsion of $H_1(N)^u$.

**Proof**

The Lemma is standard in finite transformation group theory (see, for example, [B, p. 119-120]). It follows from a transfer argument [B, p. 119, 2.2] that there exist two homomorphisms $\pi_* : H_1(N)^u \to H_1(N_u)$ and $\mu_* : H_1(N_u) \to H_1(N)^u$ such that the two maps

$$\pi_* \mu_* : H_1(N_u) \to H_1(N_u)$$

$$\mu_* \pi_* : H_1(N)^u \to H_1(N)^u$$

are 'multiplication times 2'. The Lemma follows from the remark that 'multiplication times 2' is an isomorphism when restricted to the odd torsion of $H_1(N_u)$ and $H_1(N)^u$. A similar kind of argument proves that the ranks of $H_1(N_u)$ and $H_1(N)^u$ are equal.

This finishes the proof.

By an $n$-fold branched covering of a link $L$ in the 3-sphere $S^3$, we mean a $n$-fold branched covering of $S^3$ branched along the link $L$. We use the notation $M \to S^3(L)$.

By applying the Lemma to the case that $N_u$ is the 3-sphere $S^3$, we get the following:

**Proposition 1**

Let $N$ be the 2-fold branched covering of a link $L$ in $S^3$ and $u$ the covering involution of the covering $N \to S^3(L)$. Then the involution $u$ lifts to any regular unbranched cyclic
covering \( M \) of odd order of \( N \). Moreover the set of all lifts of \( u \) to \( M \) generate a dihedral group of order \( 2n \).

**Proof**

It follows from the Lemma and the fact that the underlying topological space of the quotient \( N/u \) is \( S^3 \), that the subgroup \( H_1(N)^u \) of elements of \( H_1(N) \) which are fixed by the induced action of \( u \) has trivial odd torsion and rank zero. Any regular odd order cyclic covering of \( N \) corresponds to an epimorphism \( \psi \) of \( H_1(N) \) onto a cyclic group of odd order (so the existence of such a covering implies that either the rank \( r \) or the odd torsion \( T_2' \) of \( H_1(N) \) is not trivial). We will show that the kernel of \( \psi \) is invariant under the action of \( u \).

The kernel of \( \psi \) contains the 2-torsion \( T_2 \) (which is invariant under the action of \( u \)). Suppose that the element \( t \in \text{kernel} \psi \) has infinite order. If \( u \) maps \( t \) to \( t' \) then it maps \( t' \) to \( t \) (because \( u \) has order two), and hence \( tt' \) is fixed by \( u \). As \( u \) has no fixed points of infinite or odd order, it follows easily that \( tt' \) is in the 2-torsion \( T_2 \) of \( H_1(N) \), and consequently also \( t' \) is in the kernel of \( \psi \). Finally, since \( u \) fixes no non-trivial element of the odd torsion \( T_2' \) (which is also invariant under the action of \( u \)), it is easy to see that \( u \) sends each element of \( T_2' \) to its inverse, and in particular leaves invariant every subgroup of \( T_2' \). It follows that the kernel of \( \psi \) is invariant under the action of \( u \), and consequently \( u \) lifts to \( M \).

Let now \( \mathbb{Z}_n \) be the covering group of a regular odd order cyclic covering \( M \) of \( N \). The covering group \( \mathbb{Z}_n \) is isomorphic to the quotient of \( H_1(N) \) by the kernel of \( \psi \), and it follows from the above that a lift of \( u \) to \( M \), acting on \( \mathbb{Z}_n \) by conjugation, sends each element of \( \mathbb{Z}_n \) to its inverse. It follows that the lifts of \( u \) to \( M \) generate a dihedral group \( \mathbb{D}_n \) of order \( 2n \).

This finishes the proof of Proposition 1.

We shall need a result which estimates the Heegaard genus of a 3-manifold occurring as a branched covering of a link \( L \) in \( S^3 \) in terms of \( n \) and the bridge number of the link \( L \).

Let \( M \) be a closed orientable 3-manifold. Recall that a pair \((H_g, H'_g)\) of handlebodies of genus \( g \) is called a *Heegaard splitting of genus \( g \)* of \( M \) if \( M = H_g \cup H'_g \) and \( H_g \cap H'_g = \partial H_g = \partial H'_g \) is a closed orientable surface of genus \( g \). The minimal genus among the genera of all Heegaard splittings of \( M \) is called the *Heegaard genus of \( M \)*. The 3-sphere \( S^3 \) is the only 3-manifold of Heegaard genus zero. The Heegaard genus of \( M \) is equal to one if and only if \( M \) is a lens space \( L(p, q) \) or \( S^2 \times S^1 \); it is natural to consider these manifolds of Heegaard genus one as 3-dimensional analogues of the Riemann surface of genus one, i.e. the torus \( T^2 \).

Recall that an *\( m \)-bridge presentation* of a link \( L \) in \( S^3 \) is a decomposition of the pair \((S^3, L)\) into a union \((B_1, \alpha_1) \cup (B_2, \alpha_2)\) where \( B_i \) is a 3-ball and \( \alpha_i \) is a set of \( m \) arcs
which is trivial in $B_i$, for $i = 1, 2$. We say that $L$ is an $m$-bridge link if $m$ is the minimal number for which $L$ admits an $m$-bridge presentation.

The following is a special case of a more general result relating Heegaard splittings and bridge numbers for arbitrary branched coverings of links, see e.g. [BZ, page 169, Proposition 11.3].

**Proposition 2**

A 3-manifold which is a 3-fold branched covering of an $m$-bridge link $L$ in $S^3$, with branching index two at each component of $L$, has a Heegaard splitting of genus $m - 2$.

Finally we need also the following (see ([S, Sublemma 15.4] or [BZ, page 135, E 9.5])

**Proposition 3**

The first homology $H_1(M, \mathbb{Z}_2)$ of the 2-fold branched covering $M$ of an $r$-component link in $S^3$ is isomorphic to $(\mathbb{Z}_2)^{r-1}$.

3. Proof of Theorem 1

i) The regular case

We recall that $W_2$, being a genus two 3-manifold, admits a hyperelliptic involution, say $\tau$; the underlying topological space of the quotient $W_2/\tau$ is $S^3$, and the singular set (branch set) is a 3-bridge link $L$ (see [V]). Let $W$ be a 3-fold regular unbranched covering of $W_2$.

By Proposition 1, $\tau$ lifts to an involution $t$ of $W$, and the group generated by $t$ and the covering group $\mathbb{Z}_3$ of the covering $W \rightarrow W_2$ is isomorphic to the dihedral group $\mathbb{D}_3$ of order six (we remark that the transfer argument used for the proof of the Lemma and Proposition 1 can be avoided; we will give an alternative proof at the end of case i).

The quotient $W/t$ is a non-regular 3-fold covering of $S^3(L)$ (that is of $S^3$ branched along the link $L$). To complete the proof it is enough to show that the branching index of this non-regular covering is two at each point of $L$. Case i) of Theorem 1 follows then from Proposition 2 and the fact that $L$ is a 3-bridge link.

To compute branching indices we study the fixed points of the involutions of $\mathbb{D}_3$ in $W$.

First of all the fixed point set of $\tau$ in $W_2$ is the preimage $\bar{L}$ of $L$. Each component of $\bar{L}$ lifts to three distinct components in $W$ which are permuted by the action of the covering group $\mathbb{Z}_3$: in fact, denoting by $g$ a generator of $\mathbb{Z}_3$, the three lifts of $\tau$ to $W$ are the three conjugate involutions $t, gtg^{-1}$ and $g^2tg^{-2}$. Being conjugate, the three involutions have homeomorphic fixed point sets which are permuted by the action of $\mathbb{Z}_3$. The fixed point set of any of the three involutions projects onto $\bar{L}$ in $W_2$ and onto $L$ in $S^3$.

When factoring $W$ by $t$, the fixed point set of $t$ projects to the branch set of the covering $W \rightarrow W/t$, and the fixed point set of both $gtg^{-1}$ and $g^2tg^{-2}$ (which are conjugate by $t$)
projects to the branching set of the non-regular covering \( W/t \to S^3(L) \), with branching index two at each point.

Finally we indicate an alternative proof that \( \tau \) can be lifted to \( W \).

As above, let \( \tau \) be the hyperelliptic involution of \( W_2 \), and let \( \tau^* \) be the automorphism of \( \pi_1(W_2) \) induces by \( \tau \).

As the 3-fold covering \( W \to W_2 \) in regular, the fundamental group \( \pi_1(W) \) is the kernel of some epimorphism \( \varphi : \pi_1(W_2) \to \mathbb{Z}_3 \). We will show that \( \pi_1(W) = \text{Ker} \varphi \) is invariant under the action of \( \tau^* \).

The singular set of the orbifold \( S^3(L) = W_2/\tau \) is a 3-bridge link \( L \). Therefore, the orbifold fundamental group (see [T]) \( \pi_1^{\text{orb}}(S^3(L)) = \langle e_1, e_2, e_3 \rangle \) is generated by three involutions \( e_1, e_2, \) and \( e_3 \), corresponding to meridian loops of \( L \). The fundamental group \( \pi_1(W_2) \) is a subgroup of index two in the group \( \langle e_1, e_2, e_3 \rangle \) consisting of all words of even length in these generators; in particular, \( \pi_1(W_2) \) is generated by the elements \( A = e_1e_2 \) and \( B = e_1e_3 \). Note that \( e_2e_3 = e_2e_1 \cdot e_1e_3 = A^{-1}B \). Up to an inner automorphism, the action of \( \tau^* \) on \( \pi_1(W) \) is given by the following rule:

\[
\tau^* : A \to e_1^{-1}Ae_1 = A^{-1}, \quad B \to e_1^{-1}Be_1 = B^{-1}.
\]

We will show that \( \tau^* \) induces an automorphism of the group \( \mathbb{Z}_3 = \langle c \mid c^3 = 1 \rangle \), with respect to the epimorphism \( \varphi : \pi_1(W_2) \to \mathbb{Z}_3 \). In fact, up to a choice of notations, there are three possibilities for the action of the epimorphism \( \varphi \):

\[
(i) \quad \varphi(A) = c, \quad \varphi(B) = c,
(ii) \quad \varphi(A) = c, \quad \varphi(B) = c^{-1},
(iii) \quad \varphi(A) = c, \quad \varphi(B) = 1.
\]

Then it is clear that \( \tau^* \) induces the automorphism \( c \to c^{-1} \) of \( \mathbb{Z}_3 \), and consequently \( \text{Ker} \varphi \) is invariant under the action of the automorphism \( \tau^* \). Thus, the involution \( \tau \) lifts to an involution \( t \) of \( W \).

The group of covering transformations \( G \) of the regular 6-fold covering \( W \to W_2/\tau = S^3(L) \) has order 6 and hence is cyclic or dihedral. As the orbifold fundamental group \( \pi_1^{\text{orb}}(S^3(L)) \) is generated by three involutions and \( \pi_1^{\text{orb}}(S^3(L))/\pi_1(W) \cong G \), also \( G \) is generated by three involutions, and therefore \( G \) is a dihedral group of order six.

ii) The non-regular case

Since \( W_2 \) admits a non-regular 3-fold unbranched covering, its fundamental group \( \pi_1(W_2) \) contains a non-normal subgroup \( H \) of index three which is the fundamental group of the 3-fold covering \( W \). The action of the elements of \( \pi_1(W_2) \) on the three cosets of \( H \) in \( \pi_1(W_2) \) determines an epimorphism \( \phi : \pi_1(W_2) \to S_3 \) of \( \pi_1(W_2) \) onto the permutation group \( S_3 \) of three letters. The kernel of \( \phi \) is a normal subgroup of \( \pi_1(W_2) \).
and it corresponds to a regular $S_3$-covering of $W_2$ which we denote by $M$. See Figure 1 for the diagram of coverings constructed in the following, where ”3” corresponds to a regular 3-fold covering and ”(3)” to a non-regular one.

![Diagram of coverings](image)

Figure 1.

The covering group of the covering $M \to W_2$ is isomorphic to $S_3$ and contains three conjugate involutions; factoring $M$ by the action of one of these involutions we get (a manifold homeomorphic to) $W$. Factoring $M$ by the action of the unique subgroup $Z_3$ of $S_3$, we get a manifold, say $N$, which is a regular unbranched 2-fold covering of $W_2$.

We have recalled above in case i) that $W_2$, being a genus two 3-manifold, admits an involution $\tau$ such that the underlying topological space of the quotient $W_2/\tau$ is $S^3$ and its singular set a 3-bridge link $L$. Since $W_2$ admits the regular 2-fold unbranched covering $N$, its homology $H_1(W_2, \mathbb{Z}_2)$ is not trivial. By Proposition 3, the link $L$ has at least two components. As $L$ is a 3-bridge link, it has two or three components, and in particular $L$ is the disjoint union $L = L_0 \cup L_1$ where $L_0$ is the unknot (a 1-bridge link) and $L_1$ is a 2-bridge link with one or two components.

By Proposition 3, the homology $H_1(W_2, \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In the first case, there is exactly one 2-fold unbranched covering of $W_2$ which consequently is $N$, and the involution $\tau$ lifts to an involution of $N$. In the second case, there are exactly three 2-fold unbranched coverings of $W_2$ which have been described in [MR]. It follows from [MR, Proof of the Theorem, case ii] that the involution $\tau$ lifts to each of these three coverings of $W_2$, and hence in particular to $N$.

Thus in any case the involution $\tau$ lifts to $N$ and the manifold $N$ is a regular $\mathbb{Z}_2 \times \mathbb{Z}_2$-covering of $L$: the covering group of the covering $N \to S^3(L)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and contains three distinct involutions. It is proved in [MR, Proof of the Theorem] that exactly one of these three involutions, say $v$, acts freely on $N$; this is the covering involution of the unbranched covering $N \to W_2$. The fixed point set of a second involution, say $u$, is the preimage $\tilde{L}_1$ of $L_1$ in $N$, and the fixed point set of
the product $uv$ is the preimage $\tilde{L}_0$ of $L_0$ in $N$. The underlying topological space of the quotient $N/u$ is the 3-sphere $S^3$ because $N/u$ is the 2-fold branched covering of the trivial knot $L_0$ in $S^3$.

The manifold $M$ is a regular unbranched 3-fold cyclic covering of $N$. By construction, the free involution $v$ of $N$ lifts to $M$ and the group generated by all lifts of $v$ to $M$ is isomorphic to $\mathbb{Z}_3 \cong D_3$. As the underlying topological space of $N/u$ is $S^3$, by Proposition 1 also the involution $u$ lifts to $M$ and the group generated by all lifts of $u$ to $M$ is also isomorphic to $D_3$. This determines the structure of the group $E$ generated by all lifts of $u$ and $v$ to $M$. The group $E$ has order twelve since it contains a normal subgroup $\mathbb{Z}_3$ of index four. A Sylow 2-subgroup $S_2$ of $E$ has order four and contains two involutions conjugate, respectively, to a lift $U$ of $u$ and a lift $V$ of $v$, so the Sylow 2-subgroup of $E$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and its action on $\mathbb{Z}_3$ is determined by the dihedral actions of $U$ and $V$ on $\mathbb{Z}_3$. It follows that $E \cong S_3 \oplus \mathbb{Z}_2$, and we can assume that $UV$ has order two.

By construction, the quotient $M/V$ is the 3-fold non-regular unbranched covering $W$ of $W_2$. The involution $U$ descends to an involution $\bar{u}$ of $W$; so $W/\bar{u}$ is a 3-fold non-regular covering of $L$. To complete the proof we will show that the branching index of the non-regular covering $W/\bar{u} \to S^3(L)$ is one on the component $L_0$ of $L$ and two on (each component of) $L_1$, so $W/\bar{u}$ is a 3-fold branched covering of the 2-bridge link $L_1$. Now Proposition 2 implies that $W/\bar{u}$ has Heegaard genus zero and hence is the 3-sphere, so $W$ is hyperelliptic.

Like in the regular case in order to compute branching indices we will study the fixed point sets of the involutions.

We have seen above that the preimage $\tilde{L}$ of $L$ in $N$ splits as $\tilde{L}_0 \cup \tilde{L}_1$ where $\tilde{L}_1$ is the fixed point set of $u$ and $\tilde{L}_0$ the fixed point set of $uv$. When lifting to $M$ the lifts of $u$ and $uv$ behave in different ways.

Denoting by $g$ a generator of the covering group $\mathbb{Z}_3$ of the covering $M \to N$, the lifts of $u$ to $M$ are three distinct involutions $U$, $gUg^{-1}$ and $g^2Ug^{-2}$ which are conjugated by the action of $\mathbb{Z}_3$. So $U$, $gUg^{-1}$ and $g^2Ug^{-2}$ have homeomorphic fixed point sets which are permuted by the action of $\mathbb{Z}_3$. Each of these fixed point sets projects onto $\tilde{L}_1$ in $N$ and onto $L_1$ in $S^3$.

On the other hand there is exactly one lift $UV$ of $uv$ which is an involution (and has non-empty fixed point set) because the lifts of $uv$ to $M$ generate a cyclic group $\mathbb{Z}_6$. So the fixed point set of $UV$ is the full preimage of $\tilde{L}_0$ in $M$.

When factoring $M$ by the group $S_2$ generated by $U$ and $V$, the union of the fixed point set of $UV$ and the fixed point set of $U$ project to the branching set of the covering $M \to W/\bar{u}$. The fixed point sets of $gUg^{-1}$ and $g^2Ug^{-2}$ (which are conjugate by $U$ or $V$) project to the branching set of the non-regular covering $W/\bar{u} \to S^3(L)$ which is therefore $L_1$ with branching index two.
As noted above, an application of Proposition 2 finishes now the proof of the Theorem.

References


