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ELEMENTARY FORMULAS FOR A HYPERBOLIC TETRAHEDRON

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Abstract: We derive some elementary formulas expressing the relation between the dihedral angles and edge lengths of a tetrahedron in hyperbolic space.

Keywords: hyperbolic tetrahedron, n-dimensional hyperbolic simplex, law of sines, law of cosines

1. Introduction

Elementary formulas relating the dihedral angles and edge lengths of a tetrahedron in hyperbolic space are important in solving the classical problem of computation of the volume of a hyperbolic tetrahedron which was solved recently in [1–4]. Among the results of the present article, for instance, Theorem 2 ("The Law of Sines") we would single out as it reads classical. In a slightly different form this theorem can be found in Coolidge's book [5] written in the beginning of the last century. Theorem 4 ("The Law of Cosines") appears to us to be a new or at least well forgotten result. Both theorems were used in [4] and [6] for calculating the volume of a symmetric hyperbolic tetrahedron. Basing on these theorems, we answer in the affirmative the question of Buser concerning the relation between the face areas and heights in a hyperbolic tetrahedron. In addition, this article offers a hyperbolic analog of the generalized sine theorem (Theorem 7) obtained by Rivin [7] in the case of a Euclidean space. Our proof of that result is based on the formulas for the edge lengths and heights of an n-dimensional hyperbolic simplex also established herein.

2. Preliminaries

Following Ratcliffe [8], we recall some well-known facts of hyperbolic geometry which will be needed later. The real vector space $\mathbb{R}^{n,1}$ of dimension $n+1$ with the Lorentz inner product $\langle \mathbf{x}, \mathbf{y} \rangle = -x_0 y_0 + x_1 y_1 + \dots + x_n y_n$, where $\mathbf{x} = (x_0, x_1, \dots, x_n)$ and $\mathbf{y} = (y_0, y_1, \dots, y_n)$, is called an $(n+1)$-*dimensional Lorentzian space* $\mathbb{E}^{1,n}$.

Consider the two-sheeted hyperboloid $\mathscr{H}_t = \{\mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1\}$ and its upper sheet $\mathscr{H}_t^+ = \{\mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1,\ x_0 > 0\}$. The restriction of the quadratic form induced by the Lorentz inner product $\langle \cdot, \cdot \rangle$ to the tangent space to $\mathscr{H}_t^+$ is positive definite, and so it gives a Riemannian metric on $\mathscr{H}_t^+$. The space $\mathscr{H}_t^+$, equipped with this metric, is called a *hyperbolic model* of the n-dimensional hyperbolic space and denoted by $\mathbb{H}^n$. The hyperbolic distance d between two points $\mathbf{x}$ and $\mathbf{y}$ in this metric is given by the formula $\langle \mathbf{x}, \mathbf{y} \rangle = -\cosh d$.

Consider the cone $\mathscr{K} = \{\mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0\}$ and its upper half $\mathscr{K}^+ = \{\mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0, x_0 > 0\}$. A ray in $\mathscr{K}^+$ issuing from the origin corresponds to a point on the ideal boundary of $\mathbb{H}^n$. The set of such rays forms a sphere at infinity $\mathbf{S}^{n-1}_\infty$. Thus, each ray in $\mathscr{K}^+$ becomes an infinitely distant point of $\mathbb{H}^n$.

Denote by $\mathscr{P}$ the radial projection of $\mathbb{E}^{1,n} \backslash \{\mathbf{x} \in \mathbb{E}^{1,n} \mid x_0 = 0\}$ onto the affine hyperplane $\mathbb{P}^n_1 = \{\mathbf{x} \in \mathbb{E}^{1,n} \mid x_0 = 1\}$ along a ray issuing from the origin o. The projection $\mathscr{P}$ is a homeomorphism of $\mathbb{H}^n$ onto the open n-dimensional unit ball $\mathbf{B}^n$ in $\mathbb{P}^n_1$ centered at $(1, 0, 0, \dots, 0)$ which defines a *projective model* of $\mathbb{H}^n$. The affine hyperplane $\mathbb{P}^n_1$ includes not only $\mathbf{B}^n$ and its set-theoretic boundary $\partial \mathbf{B}^n$ in $\mathbb{P}^n_1$,

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which is canonically identified with $\mathbf{S}^{n-1}_\infty$, but also the exterior of the compactified projective model $\overline{\mathbf{B}^n} = \mathbf{B}^n \cup \partial\mathbf{B}^n \approx \mathbb{H}^n \cup \mathbf{S}^{n-1}_\infty$. Therefore, $\mathscr{P}$ can be naturally extended to a map from $\mathbb{E}^{1,n}\backslash\{0\}$ onto an n-dimensional real projective space $\mathbb{P}^n = \mathbb{P}^n_1 \cup \mathbb{P}^n_\infty$, where $\mathbb{P}^n_\infty$ is the set of straight lines in the affine hyperplane $\{\mathbf{x} \in \mathbb{E}^{1,n} \mid x_0 = 0\}$ passing through the origin. Denote by $\operatorname{Ext}\overline{\mathbf{B}^n}$ the exterior of $\overline{\mathbf{B}^n}$ in $\mathbb{P}^n$.

Consider the one-sheeted hyperboloid $\mathscr{H}_s = \{\mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{x}\rangle = 1\}$. Given some point $\mathbf{u}$ in $\mathscr{H}_s$ define in $\mathbb{E}^{1,n}$ the half-space $\mathrm{R}_\mathbf{u} = \{\mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{u}\rangle \le 0\}$ and the hyperplane $\mathrm{P}_\mathbf{u} = \{\mathbf{x} \in \mathbb{E}^{1,n} \mid \langle \mathbf{x}, \mathbf{u}\rangle = 0\} = \partial\mathrm{R}_\mathbf{u}$. Denote by $\Gamma_\mathbf{u}$ (respectively $\Pi_\mathbf{u}$) the intersection of $\mathrm{R}_\mathbf{u}$ (respectively $\mathrm{P}_\mathbf{u}$) with $\mathbf{B}^n$. Then $\Pi_\mathbf{u}$ is a geodesic hyperplane in $\mathbb{H}^n$, and the correspondence between the points in $\mathscr{H}_s$ and the half-space $\Gamma_\mathbf{u}$ in $\mathbb{H}^n$ is bijective. Call the vector $\mathbf{u}$ *normal to* $\mathrm{P}_\mathbf{u}$ (or $\Pi_\mathbf{u}$).

Proposition 1. *Take two noncollinear points* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathscr{H}_s$. *One of the following holds:*

(i) *The geodesic hyperplanes* $\Pi_\mathbf{x}$ *and* $\Pi_\mathbf{y}$ *intersect provided that* $|\langle \mathbf{x}, \mathbf{y}\rangle| < 1$. *In this case the (hyperbolic) angle* θ *between them is given by* $\langle \mathbf{x}, \mathbf{y}\rangle = -\cos\theta$.

(ii) *The geodesic hyperplanes* $\Pi_\mathbf{x}$ *and* $\Pi_\mathbf{y}$ *do not intersect in* $\overline{\mathbf{B}^n}$; *thus, they intersect in* $\operatorname{Ext}\overline{\mathbf{B}^n}$ *provided that* $|\langle \mathbf{x}, \mathbf{y}\rangle| > 1$. *In this case the (hyperbolic) distance* d *between them is given by* $|\langle \mathbf{x}, \mathbf{y}\rangle| = -\cosh d$. *Such* $\Pi_\mathbf{x}$ *and* $\Pi_\mathbf{y}$ *are called ultraparallel.*

(iii) *The geodesic hyperplanes* $\Pi_\mathbf{x}$ *and* $\Pi_\mathbf{y}$ *do not intersect in* $\mathbf{B}^n$ *but intersect in* $\partial\mathbf{B}^n$ *provided that* $|\langle \mathbf{x}, \mathbf{y}\rangle| = 1$. *In this case the angle and distance between them are both zero. Such* $\Pi_\mathbf{x}$ *and* $\Pi_\mathbf{y}$ *are called parallel.*

Proposition 2. *Take a point* $\mathbf{x}$ *in* $\mathbf{B}^n$ *and a geodesic hyperplane* $\Pi_\mathbf{y}$ *whose normal vector* $\mathbf{y}$ *lies in* $\mathscr{H}_s$ *so that* $\langle \mathbf{x}, \mathbf{y}\rangle < 0$. *Then the distance* d *between* $\mathbf{x}$ *and* $\Pi_\mathbf{y}$ *is given by* $\langle \mathbf{x}, \mathbf{y}\rangle = -\sinh d$.

Take $\mathbf{v} \in \operatorname{Ext}\overline{\mathbf{B}^n}$. Then $\mathscr{P}^{-1}(\mathbf{v}) \cap \mathscr{H}_s$ contains two points that define the same hyperplane $\Pi_{\tilde{\mathbf{v}}}$, $\tilde{\mathbf{v}} \in \mathscr{P}^{-1}(\mathbf{v}) \cap \mathscr{H}_s$. Call $\Pi_{\tilde{\mathbf{v}}}$ the *polar geodesic hyperplane to* $\mathbf{v}$, and call $\mathbf{v}$ the *pole* of $\Pi_{\tilde{\mathbf{v}}}$.

Proposition 3. *Take* $\mathbf{v} \in \operatorname{Ext}\overline{\mathbf{B}^n}$.

(i) *Every hyperplane passing through* $\mathbf{v}$ *and crossing* $\mathbf{B}^n$ *is orthogonal to* $\Pi_{\tilde{\mathbf{v}}}$ *in* $\mathbb{H}^n$.

(ii) *If* $\mathbf{u} \in \mathscr{P}_{\tilde{\mathbf{v}}} \cap \partial\mathbf{B}^n$ *then the line through* $\mathbf{u}$ *and* $\mathbf{v}$ *is tangent to* $\partial\mathbf{B}^n$.

3. The n-Dimensional Generalized Hyperbolic Simplex

Assume now that $n \ge 3$. Denote by Δ a convex polyhedron in $\mathbb{P}^n_1$. Assume that each of the edges (the $(n-2)$-dimensional faces) of Δ crosses $\overline{\mathbf{B}^n}$.

Definition 1. Take a vertex $\mathbf{v}$ of Δ, with $\mathbf{v} \in \operatorname{Ext}\overline{\mathbf{B}^n}$. Call a *truncation* of Δ the operation of removing the pyramid with apex $\mathbf{v}$ and base $\Pi_{\tilde{\mathbf{v}}} \cap \Delta$; call the *truncated polyhedron* Δ' the polyhedron obtained by truncating all vertices lying in $\operatorname{Ext}\overline{\mathbf{B}^n}$. Call $\mathbf{v}$ a *finite, ideal*, and *ultraideal* vertex of Δ' in the cases that $v \in \mathbf{B}^n$, $\partial\mathbf{B}^n$, and $\operatorname{Ext}\overline{\mathbf{B}^n}$ respectively. By a *generalized hyperbolic polyhedron* we will mean either a polyhedron in the usual sense or a truncated polyhedron.

Consider the n-dimensional generalized hyperbolic simplex σ^n with vertices v_i, heights h_i and edge lengths l_{ij}, $i, j = 1, 2, \dots, n+1$, in the projective model $\mathbf{B}^n$ of the hyperbolic space. Denote by $G = \langle -\cos\alpha_{ij}\rangle_{i,j=1,\dots,n+1}$ the Gram matrix of σ^n, where α_{ij} is the dihedral angle at the ij-face of the simplex, which is opposite to v_i and v_j. Denote by $C = \langle c_{ij}\rangle_{i,j=1,\dots,n+1}$ the adjoint matrix that consists of the elements $c_{ij} = (-1)^{i+j} G_{ij}$, where G_{ij} is the ijth minor of G. The following theorem gives necessary and sufficient conditions for the existence of a generalized hyperbolic simplex in terms of its Gram matrix.

Theorem 1 [3]. *Given a set* $\{\theta_{ij} \in [0, \pi] \mid i, j = 1, \dots, n+1,\ \theta_{ij} = \theta_{ji},\ \theta_{ii} = \pi,\ i = j\}$ *of positive real numbers, the following two conditions are equivalent:*

(1) *there exists a generalized hyperbolic simplex in* $\mathbb{H}^n$ *with the dihedral angle between the* i*th face and the* j*th face equal to* θ_{ij};

(2) *the real symmetric matrix* $G = \langle -\cos\theta_{ij}\rangle_{i,j=1,\dots,n+1}$ *of size* $n+1$ *satisfies the following conditions:*

(i) $\operatorname{sgn} G = (n, 1)$; *i.e.,* G *has one negative and* n *positive eigenvalues;*

(ii) $c_{ij} > 0$, $i \ne j$, $i, j = 1, 2, \dots, n+1$.

The ith vertex of the simplex is finite, ideal, or ultraideal provided that $c_{ii} > 0$, $c_{ii} = 0$, or $c_{ii} < 0$ respectively.

Proposition 4. *Take an n-dimensional generalized hyperbolic simplex σ^n with finite vertices v_i, heights h_i, and edge lengths $l_{ij}, i, j = 1, 2, \dots, n+1$. Then*

$$\cosh l_{ij} = \frac{c_{ij}}{\sqrt{c_{ii}c_{jj}}}, \tag{i}$$

$$\sinh h_j = \frac{\sqrt{-\det G}}{\sqrt{c_{jj}}}. \tag{ii}$$

PROOF. Theorem 1 implies that $\det G < 0$ for the Gram matrix

$$G = \langle g_{ij} \rangle_{i,j=1,\dots,n+1} = \langle -\cos\theta_{ij} \rangle_{i,j=1,\dots,n+1}$$

of a hyperbolic simplex σ^n. Then there exists a basis of $n+1$ unit vectors $\{u_1, \dots, u_{n+1}\}$ in $\mathbb{E}^{1,n}$ such that $\langle u_i, u_j \rangle = -\cos\theta_{ij}$. Consider the system of vectors $\{w_1, \dots, w_{n+1}\}$ with $w_i = \sum_{k=1}^{n+1} c_{ik}u_k$, where c_{ik} are the elements of the adjoint matrix C. We have

$$\langle w_i, u_j \rangle = \left\langle \sum_{k=1}^{n+1} c_{ik}u_k, u_j \right\rangle = \sum_{k=1}^{n+1} c_{ik}\langle u_k, u_j \rangle = \sum_{k=1}^{n+1} c_{ik}g_{kj} = \delta_{ij} \det G,$$

where δ_{ij} is the Kronecker symbol. Thus, the system of vectors $\{w_1, \dots, w_{n+1}\}$ defines the basis in $\mathbb{E}^{1,n}$ dual to $\{u_1, \dots, u_{n+1}\}$. In order to make it orthonormal, compute

$$\langle w_i, w_j \rangle = \left\langle \sum_{k=1}^{n+1} c_{ik}u_k, w_j \right\rangle = \sum_{k=1}^{n+1} c_{ik}\langle u_k, w_j \rangle = \sum_{k=1}^{n+1} c_{ik}\delta_{kj} \det G = c_{ij} \det G$$

and denote by $v_i = \frac{w_i}{\sqrt{-c_{ii}\det G}}$, with $c_{ii} > 0$, the vectors of the orthonormal basis. This basis defines some system of vectors at the vertices of the simplex. We have

$$\langle v_i, v_j \rangle = \frac{-c_{ij}}{\sqrt{c_{ii}c_{jj}}}, c_{ii}c_{jj} > 0.$$

On the other hand, $\langle v_i, v_j \rangle = -\cosh l_{ij}$; see Proposition 1 (ii). Consequently, $\cosh l_{ij} = \frac{c_{ij}}{\sqrt{c_{ii}c_{jj}}}$.

Furthermore, the biorthogonality of u_k and v_j on the one hand yields

$$\langle w_i, v_j \rangle = \sum_{k=1}^{n+1} c_{ik}\langle u_k, v_j \rangle = c_{ij}\langle u_j, v_j \rangle.$$

On the other hand,

$$\langle w_i, v_j \rangle = \langle \sqrt{-c_{ii}\det G} v_i, v_j \rangle = \sqrt{-c_{ii}\det G}\frac{-c_{ij}}{\sqrt{c_{ii}c_{jj}}} = -c_{ij}\frac{\sqrt{-\det G}}{\sqrt{c_{jj}}}.$$

Therefore, we obtain

$$-c_{ij}\frac{\sqrt{-\det G}}{\sqrt{c_{jj}}} = c_{ij}\langle u_j, v_j \rangle$$

or

$$\langle u_j, v_j \rangle = -\sqrt{\frac{-\det G}{c_{jj}}}.$$

Proposition 2 implies that $\langle u_j, v_j \rangle = -\sinh h_j$. Hence,

$$\sinh h_j = \frac{\sqrt{-\det G}}{\sqrt{c_{jj}}}. \qquad \square$$

REMARK 1. (1) If v_i is a finite vertex of a generalized simplex and v_j is an ultraideal vertex then $\cosh l_{ij} = \frac{-c_{ij}}{\sqrt{-c_{ii}c_{jj}}}$.

(2) If v_i and v_j are ultraideal vertices of a generalized simplex then $\cosh l_{ij} = \frac{-c_{ij}}{\sqrt{c_{ii}c_{jj}}}$.

4. The Laws of Sines and Cosines for a Hyperbolic Tetrahedron

Consider a tetrahedron $T = T(A, B, C, D, E, F) \in \mathbb{H}^3$ with heights h_1, h_2, h_3, h_4, corresponding to the vertices v_1, v_2, v_3, v_4 respectively, and dihedral angles A, B, C, D, E, F at the edges of lengths l_A, l_B, l_C, l_D, l_E, l_F respectively, as depicted in Fig. 1.

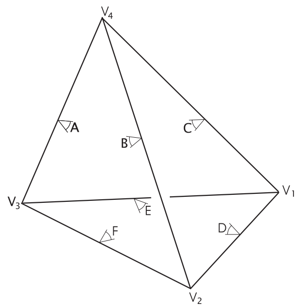

Fig. 1

The necessary and sufficient conditions for the existence of a tetrahedron in hyperbolic space are given in Theorem 1 in terms of the Gram matrix. The tetrahedron is defined uniquely up to an isometry by the collection of its dihedral angles. Denote the Gram matrix of T by

$$G = \langle -\cos\theta_{ij}\rangle_{i,j=1,2,3,4} = \begin{pmatrix} 1 & -\cos A & -\cos B & -\cos F \\ -\cos A & 1 & -\cos C & -\cos E \\ -\cos B & -\cos C & 1 & -\cos D \\ -\cos F & -\cos E & -\cos D & 1 \end{pmatrix},$$

and the adjoint matrix consisting of the elements $c_{ij} = (-1)^{i+j} G_{ij}$, where G_{ij} is the ijth minor of G, by $C = \langle c_{ij}\rangle_{i,j=1,2,3,4}$.

Take the conjugate Gram matrix

$$G^* = \langle v_i, v_j\rangle_{i,j=1,2,3,4} = \begin{pmatrix} -1 & -\cosh l_D & -\cosh l_E & -\cosh l_C \\ -\cosh l_D & -1 & -\cosh l_F & -\cosh l_B \\ -\cosh l_E & -\cosh l_F & -1 & -\cosh l_A \\ -\cosh l_C & -\cosh l_B & -\cosh l_A & -1 \end{pmatrix}$$

consisting of the elements

$$\langle v_i, v_j\rangle = \frac{-c_{ij}}{\sqrt{c_{ii}c_{jj}}}, \quad i, j = 1, 2, 3, 4;$$

see Proposition 4 (i) and Fig. 1. Put $c^*_{ij} = (-1)^{i+j} G^*_{ij}$, where G^*_{ij} is the ijth minor of G^*.

Proposition 5. *Take a hyperbolic tetrahedron T. The following hold:*

$$\det G^* = \frac{(\det G)^3}{P}, \quad \det G = \frac{(\det G^*)^3}{P^*}, \tag{i}$$

$$P^* = \frac{(\det G)^8}{P^3}, \quad P = \frac{(\det G^*)^8}{(P^*)^3}, \tag{ii}$$

$$\frac{P^*}{P} = \left(\frac{\det G^*}{\det G}\right)^4, \tag{iii}$$

$$\frac{\det G^*}{\det G} = \sinh h_1 \sinh h_2 \sinh h_3 \sinh h_4, \tag{iv}$$

where $P = c_{11}c_{22}c_{33}c_{44}$, $P^* = c^*_{11}c^*_{22}c^*_{33}c^*_{44}$, and h_1, h_2, h_3, h_4 are the heights of T.

Remark 2. The relations (i) in Proposition 5 enable us to express $\det G^*$ in terms of the dihedral angles of T, and $\det G$, in terms of its edge lengths.

Proof. (i) Because $C = G^{-1} \det G$,

$$\det C = \det G^{-1} (\det G)^4 = (\det G)^{-1} (\det G)^4 = (\det G)^3.$$

By the definition of G^*

$$\det G^* = \frac{1}{c_{11}c_{22}c_{33}c_{44}} \det C = \frac{(\det G)^3}{c_{11}c_{22}c_{33}c_{44}} = \frac{(\det G)^3}{P}.$$

The equality $\det G = \frac{(\det G^*)^3}{P^*}$ is verified similarly.

(ii) These equalities yield

$$P \det G^* = (\det G)^3 = \left(\frac{(\det G^*)^3}{P^*}\right)^3 = \frac{(\det G^*)^9}{(P^*)^3}.$$

Hence, $P = \frac{(\det G^*)^8}{(P^*)^3}$. The equality $P^* = \frac{(\det G)^8}{P^3}$ is verified similarly.

(iii) The ratio of the equalities $\det G^* = \frac{(\det G)^3}{P}$ and $\det G = \frac{(\det G^*)^3}{P^*}$ is

$$\frac{\det G^*}{\det G} = \frac{(\det G)^3}{P} \frac{P^*}{(\det G^*)^3},$$

which implies that

$$\frac{P^*}{P} = \frac{(\det G^*)^4}{(\det G)^4}.$$

(iv) By (i)

$$\frac{\det G^*}{\det G} = \frac{(\det G)^2}{P}.$$

On the other hand, Proposition 4 (ii) implies that

$$\sinh h_1 \sinh h_2 \sinh h_3 \sinh h_4 = \frac{(\det G)^2}{c_{11}c_{22}c_{33}c_{44}} = \frac{(\det G)^2}{P}.$$

Comparison of these expressions yields

$$\frac{\det G^*}{\det G} = \sinh h_1 \sinh h_2 \sinh h_3 \sinh h_4. \qquad \square$$

The next result was originally obtained by Coolidge [5] in a slightly different form and later reproven by Fenchel [9]. The history of the question goes back to an 1877 article of Enrico d'Ovidio, and is narrated in [10].

Theorem 2. *For a hyperbolic tetrahedron T*

$$\frac{\sin A \sin D}{\sinh l_A \sinh l_D} = \frac{\sin B \sin E}{\sinh l_B \sinh l_E} = \frac{\sin C \sin F}{\sinh l_C \sinh l_F} = \sqrt{\frac{\det G}{\det G^*}}.$$

Proof. Use the following theorem of Jacobi; cp. [11, Theorem 2.5.3].

Theorem 3 (Jacobi). *Take an $n \times n$ matrix $A = \langle a_{ij} \rangle_{i,j=1,\dots,n}$ with determinant $\det A = \Delta$. Denote by $C = \langle c_{ij} \rangle_{i,j=1,\dots,n}$ the matrix consisting of the elements $c_{ij} = (-1)^{i+j} A_{ij}$, where A_{ij} is the ijth minor of A. Then for each k with $1 \le k \le n$*

$$\det\langle c_{ij} \rangle_{i,j=1,\dots,k} = \Delta^{k-1} \det\langle a_{ij} \rangle_{i,j=k+1,\dots,n}.$$

Moreover, if $\sigma = \begin{pmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{pmatrix}$ is some permutation of $\{1, 2, \dots, n\}$ then for each k with $1 \le k \le n$

$$\det\langle c_{i_p j_q} \rangle_{p,q=1,\dots,k} = (-1)^\sigma \Delta^{k-1} \det\langle a_{i_p j_q} \rangle_{p,q=k+1,\dots,n}.$$

Applying this theorem to the Gram matrix G and its adjoint matrix C for $k = 2$ we obtain

$$c_{11}c_{22} - c_{12}^2 = (1 - \cos^2 D) \det G.$$

Similarly,

$$c_{33}c_{44} - c_{34}^2 = (1 - \cos^2 A) \det G.$$

By Proposition 4 we have $\cosh l_D = \frac{c_{12}}{\sqrt{c_{11}c_{22}}}$; thus,

$$\sinh l_D = \sqrt{\frac{c_{12}^2 - c_{11}c_{22}}{c_{11}c_{22}}}.$$

Similarly we establish that

$$\sinh l_A = \sqrt{\frac{c_{34}^2 - c_{33}c_{44}}{c_{33}c_{44}}}.$$

Hence,

$$\frac{\sin A \sin D}{\sinh l_A \sinh l_D} = \frac{\sqrt{c_{11}c_{22}c_{33}c_{44}}}{-\det G}.$$

Proposition 5 (i) implies that

$$\frac{\sqrt{c_{11}c_{22}c_{33}c_{44}}}{-\det G} = \sqrt{\frac{(\det G)^3}{\det G^*}\frac{1}{(-\det G)}} = \sqrt{\frac{\det G}{\det G^*}}. \qquad \square$$

Theorem 4. *Suppose that C, F and B, E are pairs of dihedral angles at the opposite edges of a hyperbolic tetrahedron T with lengths l_C, l_F and l_B, l_E respectively. Then*

$$\frac{\cos C \cos F - \cos B \cos E}{\cosh l_B \cosh l_E - \cosh l_C \cosh l_F} = \sqrt{\frac{\det G}{\det G^*}}.$$

Proof. Applying Theorem 3 to the matrix G for $k = 2$, we obtain the equality

$$c_{13}c_{24} - c_{14}c_{23} = (\cos B \cos E - \cos C \cos F) \det G.$$

Proposition 4 implies that $\cosh l_E = \frac{c_{13}}{\sqrt{c_{11}c_{33}}}$; hence, $c_{13} = \cosh l_E \sqrt{c_{11}c_{33}}$. We find c_{23}, c_{14}, and c_{24} similarly.

The substitution of these expressions into the previous equality yields

$$(\cos B \cos E - \cos C \cos F) \det G = \sqrt{c_{11}c_{22}c_{33}c_{44}}(\cosh l_E \cosh l_B - \cosh l_C \cosh l_F).$$

For convenience, rewrite this as

$$(\cos C \cos F - \cos B \cos E)(-\det G) = \sqrt{c_{11}c_{22}c_{33}c_{44}}(-\cosh l_C \cosh l_F + \cosh l_E \cosh l_B).$$

Hence,

$$\frac{\cos C \cos F - \cos B \cos E}{\cosh l_B \cosh l_E - \cosh l_C \cosh l_F} = \frac{\sqrt{P}}{-\det G}.$$

Proposition 5 (i) implies

$$\frac{\sqrt{P}}{-\det G} = \sqrt{\frac{\det G}{\det G^*}}.$$

The theorem is proved. $\square$

Corollary 1. *For a hyperbolic tetrahedron T we have*

$$\frac{\cos(C + \varepsilon F) - \cos(B + \delta E)}{\cosh(l_B + \delta l_E) - \cosh(l_C + \varepsilon l_F)} = \sqrt{\frac{\det G}{\det G^*}},$$

where $\varepsilon, \delta \in \{-1, 1\}$.

PROOF. By Theorem 2

$$\frac{\sin B \sin E}{\sinh l_B \sinh l_E} = \frac{\sin C \sin F}{\sinh l_C \sinh l_F} = \sqrt{\frac{\det G}{\det G^*}},$$

and by Theorem 4

$$\frac{\cos C \cos F - \cos B \cos E}{\cosh l_B \, \cosh l_E - \cosh l_C \, \cosh l_F} = \sqrt{\frac{\det G}{\det G^*}}.$$

Using the properties of ratios and the trigonometric addition/subtraction formulas, we obtain

$$\frac{\cos C \cos F - \cos B \cos E + \delta \sin B \sin E - \varepsilon \sin C \sin F}{\cosh l_B \cosh l_E - \cosh l_C \cosh l_F + \delta \sinh l_B \sinh l_E - \varepsilon \sinh l_C \sinh l_F}$$
$$= \frac{\cos(C + \varepsilon F) - \cos(B + \delta E)}{\cosh(l_B + \delta l_E) - \cosh(l_C + \varepsilon l_F)} = \sqrt{\frac{\det G}{\det G^*}}. \qquad \square$$

5. The Multidimensional Law of Sines in Hyperbolic Space

The following theorem is well known; cp. [12, p. 258].

Theorem 5 (a hyperbolic analog of Heron's formula). *The area S of a hyperbolic triangle with sides a, b, c satisfies the relation*

$$4 \sin^2 \frac{S}{2} = \frac{\sinh p \sinh(p-a) \sinh(p-b) \sinh(p-c)}{\cosh^2 \frac{a}{2} \cosh^2 \frac{b}{2} \cosh^2 \frac{c}{2}}, \tag{1}$$

where $p = \frac{a+b+c}{2}$.

Proposition 4 (ii) for the hyperbolic tetrahedron T depicted in Fig. 1 yields

$$\sinh h_1 \sqrt{c_{11}} = \sinh h_2 \sqrt{c_{22}} = \sinh h_3 \sqrt{c_{33}} = \sinh h_4 \sqrt{c_{44}} = \sqrt{-\det G}.$$

Similarly,

$$\sinh h_1 \sqrt{c_{11}^*} = \sinh h_2 \sqrt{c_{22}^*} = \sinh h_3 \sqrt{c_{33}^*} = \sinh h_4 \sqrt{c_{44}^*} = \sqrt{-\det G^*}.$$

Hence, it is not difficult to notice the equalities

$$\frac{c_{11}}{c_{11}^*} = \frac{c_{22}}{c_{22}^*} = \frac{c_{33}}{c_{33}^*} = \frac{c_{44}}{c_{44}^*} = \frac{\det G}{\det G^*}. \tag{2}$$

The direct calculation of the elements c_{ii}, c_{ii}^*, $i = 1, 2, 3, 4$, of the adjoint matrices of T and some elementary trigonometric manipulations yield the following statement.

Lemma 1. *We have*

$$\frac{\sinh p_{123} \sinh(p_{123} - l_D) \sinh(p_{123} - l_E) \sinh(p_{123} - l_F)}{\sinh p_{124} \sinh(p_{124} - l_D) \sinh(p_{124} - l_C) \sinh(p_{124} - l_B)}$$
$$= \frac{\sin \rho_{123} \sin(\rho_{123} - A) \sin(\rho_{123} - B) \sin(\rho_{123} - C)}{\sin \rho_{124} \sin(\rho_{124} - A) \sin(\rho_{124} - F) \sin(\rho_{124} - E)},$$

where $p_{123} = \frac{l_D + l_E + l_F}{2}$, $p_{124} = \frac{l_D + l_C + l_B}{2}$, $\rho_{123} = \frac{A+B+C}{2}$, and $\rho_{124} = \frac{A+F+E}{2}$.

The explicit expression for the height of T from Proposition 4 (ii) has the form

$$\sinh h_4 = \frac{\sqrt{-\det G}}{\sqrt{1 - \cos^2 A - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C}}$$
$$= \frac{\sqrt{-\det G}}{2\sqrt{\sin \rho_{123} \sin(\rho_{123} - A) \sin(\rho_{123} - B) \sin(\rho_{123} - C)}}, \tag{3}$$

where $\rho_{123} = \frac{A+B+C}{2}$.

At the international conference on analysis and geometry in honor of Academician Reshetnyak (August 30–September 3, 1999, Novosibirsk, Russia) Buser asked whether there exists a three-dimensional analog of the rule

$$\sinh a_1 \sinh h_1 = \sinh a_2 \sinh h_2 = \sinh a_3 \sinh h_3$$

relating the side lengths and heights of a hyperbolic triangle which he used in [13] for computing the spectrum of the Laplace–Beltrami operator on compact Riemann surfaces. The next statement answers his question in the affirmative.

Proposition 6. *Denote by A_1, A_2, A_3, A_4 the areas of the faces of the hyperbolic tetrahedron T opposite to the vertices v_1, v_2, v_3, v_4 respectively. Then*

$$\sin\frac{A_1}{2}\sinh h_1\cosh\frac{l_A}{2}\cosh\frac{l_B}{2}\cosh\frac{l_F}{2} = \sin\frac{A_2}{2}\sinh h_2\cosh\frac{l_A}{2}\cosh\frac{l_C}{2}\cosh\frac{l_E}{2}$$
$$= \sin\frac{A_3}{2}\sinh h_3\cosh\frac{l_B}{2}\cosh\frac{l_C}{2}\cosh\frac{l_D}{2} = \sin\frac{A_4}{2}\sinh h_4\cosh\frac{l_D}{2}\cosh\frac{l_E}{2}\cosh\frac{l_F}{2}.$$

Proof. Find the areas of the faces of T with vertices v_1, v_2, v_3 and v_1, v_2, v_4 by Theorem 5. Find its heights h_3, h_4 by (3). The substitution of the resulting expressions into the statement of Lemma 1 and some elementary manipulations yield

$$\frac{\sin\frac{A_4}{2}\cosh\frac{l_D}{2}\cosh\frac{l_E}{2}\cosh\frac{l_F}{2}}{\sin\frac{A_3}{2}\cosh\frac{l_B}{2}\cosh\frac{l_C}{2}\cosh\frac{l_D}{2}} = \frac{\sinh h_3}{\sinh h_4}. \qquad \square$$

Rivin [7] proposed the following variant of the multidimensional law of sines for an n-dimensional Euclidean simplex σ^n. In the case $n = 2$ it is equivalent to the law of sines for a Euclidean triangle.

Theorem 6 (the multidimensional law of sines). *Given an n-dimensional Euclidean simplex σ^n, for all $1 \le i, j, k, l \le n + 1$ we have*

$$\frac{A_i A_j}{A_k A_l} = \frac{c_{ij}}{c_{kl}}, \tag{4}$$

where A_i, A_j, A_k, A_l are the areas of the corresponding $(n-1)$-dimensional faces of σ^n, and $c_{ij} = (-1)^{i+j}G_{ij}$, $c_{kl} = (-1)^{k+l}G_{kl}$, where G_{ij} and G_{kl} are the ijth and klth minors of the Gram matrix.

The equality (4) has another equivalent form

$$\frac{h_k h_l}{h_i h_j} = \frac{c_{ij}}{c_{kl}}, \tag{5}$$

where $h_i, i = 1, 2, \dots, n+1$ is the height of σ^n corresponding to the ith vertex.

A hyperbolic analog of the last expression is as follows:

Theorem 7. *Given an n-dimensional hyperbolic simplex σ^n with heights h_i and edge lengths l_{ij}, $i, j = 1, 2, \dots, n$, we have*

$$\frac{\sinh h_k \sinh h_l \cosh l_{ij}}{\sinh h_i \sinh h_j \cosh l_{kl}} = \frac{c_{ij}}{c_{kl}}, \tag{6}$$

where $c_{ij} = (-1)^{i+j}G_{ij}$, and G_{ij} is the ijth minor of the Gram matrix.

Proof. The substitution of the expressions

$$\cosh l_{ij} = \frac{c_{ij}}{\sqrt{c_{ii}c_{jj}}}, \quad \sinh h_j = \frac{\sqrt{-\det G}}{\sqrt{c_{jj}}}$$

for the heights and edge lengths of the simplex of Proposition 4 on the left-hand side of (6) yields a true identity. $\square$

In the case of a tetrahedron ($n = 3$) the hyperbolic analog of Theorem 6 follows from Theorem 7 and Proposition 6.

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