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MATHEMATICS

A New Method for Counting Coverings over Manifold with Finitely Generated Fundamental Group

A. D. Mednykh

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INTRODUCTION

Let $\mathcal{M}$ be a manifold with fundamental group $\Gamma = \pi_1(\mathcal{M})$. Two coverings p_1: $\mathcal{M}_1 \to \mathcal{M}$ and p_2: $\mathcal{M}_2 \to \mathcal{M}$ are said to be equivalent if there exists a homeomorphism h: $\mathcal{M}_1 \to \mathcal{M}_2$ such that $p_1 = p_2 \circ h$. According to the general theory of covering spaces, any n-fold covering is uniquely determined by a subgroup of index n in the group Γ. Moreover, two coverings are equivalent if and only if the corresponding subgroups are conjugate in Γ. A similar assertion formulated in the language of orbifolds is valid for branched coverings.

The problem about the number of nonequivalent coverings over a Riemann surface with given branch type goes back to the paper [4] by Hurwitz, in which the number of coverings over the Riemann sphere with given number of simple (of order two) branching points was determined. Later, in [5], it was found that this number has an adequate expression in terms of irreducible characters of symmetric groups, the theory of which was developed by Frobenius in the beginning of the twentieth century. The Hurwitz problem was considered by many authors. A detailed survey of the related results is contained in [2, 9]. For closed Riemann surfaces, this problem was completely solved in [11]. However, of most interest is the case of unramified coverings. Let $M_\Gamma(n)$ denote the number of subgroups of index n in the group Γ, and let $N_\Gamma(n)$ be the number of conjugacy classes of such subgroups. According to what was said above, $N_\Gamma(n)$ coincides with the number of nonequivalent n-fold coverings over a manifold $\mathcal{M}$ with fundamental group Γ. If $\mathcal{M}$ is a compact surface with nonempty boundary of Euler characteristic $\chi(\mathcal{M}) = 1 - r$, where $r \geq 0$ (e.g., a disk with r holes), then its fundamental group $\Gamma = F_r$ is the free group of rank r. For this case, Hall calculated the number $M_\Gamma(n)$ [3] and Liskovets found the number $N_\Gamma(n)$ by using his own method for calculating the number of conjugacy classes of subgroups in free groups [6]. The numbers $M_\Gamma(n)$ and $N_\Gamma(n)$ for the fundamental group of a closed surface (orientable or not) were calculated in [10, 12].

In the three-dimensional case, for a large class of Seifert fibrations, the value of $M_\Gamma(n)$ was determined in [8]. Asymptotic formulas for $M_\Gamma(n)$ in many important cases were obtained by Müller and his coauthors [14, 15].

In this paper, we suggest a new method for calculating the number $N_\Gamma(n)$ of conjugacy classes of subgroups in any finitely generated group Γ (Theorem 1). This result implies Theorems 5–7, which provide means for counting coverings of given multiplicity over any manifold with finitely generated fundamental group. The earlier results about the number of coverings over a compact surface (with or without boundary, orientable or not) are the simplest special cases of these theorems; for completeness, we cite them in Theorems 3–5.

The methods can also be used to count nonisomorphic maps on a Riemann surface. Earlier, this problem was solved only for the sphere [7]. The method suggested in this paper makes it possible to solve it for a surface of any genus [13].

MAIN RESULT

Let Epi(K, Z_l) denote the set of all epimorphisms of the group K to the cyclic group Z_l of order l, and let $|E|$ be the cardinality of the set E.

The main result of this paper is the following theorem.

Theorem 1. *Let* Γ *be an arbitrary finitely generated group.*

Sobolev Institute of Mathematics, Siberian Division, Russian Academy of Sciences, pr. Akademika Koptyuga 4, Novosibirsk, 630090 Russia
e-mail: mednykh@math.nsc.ru

Then, the number of conjugacy classes of subgroups of index n in the group Γ *is determined by the formula*

$$N_\Gamma(n) = \frac{1}{n}\sum_{\substack{l|n \\ lm = n}}\sum_{K \underset{m}{<} \Gamma}|\mathrm{Epi}(K, Z_l)|,$$

where the sum $\sum_{K \underset{m}{<} \Gamma}$ *is taken over all subgroups K of index m in the group* Γ.

The proof of Theorem 1 relies on the following two elementary lemmas. For a subgroup P of Γ, let $N(P, \Gamma)$ denote the normalizer of P in the group Γ.

Lemma 1. *The number of conjugacy classes of subgroups of index n in the group* Γ *is determined by the formula*

$$N_\Gamma(n) = \frac{1}{n}\sum_{P \underset{n}{<} \Gamma}|N(P, \Gamma)/P|.$$

The proof of Lemma 1 is based on the following considerations. Let E be a class of conjugate subgroups of index n in Γ. Note that

$$\sum_{P \in E}|N(P, \Gamma)/P| = n.$$

Indeed, let $P' \in E$. Then, $|E| = |\Gamma : N(P', \Gamma)|$, and, for any $P \in E$, the groups $N(P, \Gamma)/P$ and $N(P', \Gamma)/P'$ are isomorphic. We have

$$\sum_{P \in E}|N(P, \Gamma)/P| = |E| \cdot |N(P', \Gamma)/P'|$$

$$= |\Gamma : N(P', \Gamma)| \cdot |N(P', \Gamma) : P'| = |\Gamma : P'| = n.$$

Therefore,

$$nN(n) = \sum_E n = \sum_E \sum_{P \in E}|N(P, \Gamma)/P|$$

$$= \sum_{P \underset{n}{<} \Gamma}|N(P, \Gamma)/P|,$$

where the sum $\sum_E$ is over all conjugacy classes E of subgroups of index n in the group Γ.

Lemma 2. *Let P be a subgroup of index n in the group* Γ. *Then*,

$$|N(P, \Gamma)/P| = \sum_{\substack{l|n \\ lm = n}}\sum_{P \underset{Z_l}{\lhd} K \underset{m}{<} \Gamma}\phi(l),$$

where $\phi(l)$ *is the Euler function and the second sum is over all subgroups K of index m in* Γ *containing P as a normal subgroup such that* $K/P \cong Z_l$. *If there are no such subgroups, then the corresponding sum is set equal to zero.*

To prove Lemma 2, we set $G = N(P, \Gamma)/P$. Since $P \lhd N(P, \Gamma) < \Gamma$ and $P \underset{n}{<} \Gamma$, it follows that the order of any cyclic subgroup of G divides n. Note that there is a one-to-one correspondence between the cyclic subgroups Z_l in G and the subgroups K satisfying the condition $P \underset{Z_l}{\lhd} K \underset{m}{<} \Gamma$, where $lm = n$. Each cyclic subgroup $Z_l < G$ contains precisely $\phi(l)$ elements of G generating Z_l. Therefore,

$$|G| = \sum_{l|n}\phi(l)\sum_{Z_l < G}1 = \sum_{l|n}\phi(l)\sum_{P \underset{Z_l}{\lhd} K \underset{m}{<} \Gamma}1$$

$$= \sum_{l|n}\sum_{P \underset{Z_l}{\lhd} K \underset{m}{<} \Gamma}\phi(l).$$

Theorem 1 is proved by applying Lemma 1 and Lemma 2 with $lm = n$. We have

$$nN(n) = \sum_{P \underset{n}{<} \Gamma}|N(P, \Gamma)/P|$$

$$= \sum_{P \underset{n}{<} \Gamma}\sum_{l|n}\sum_{P \underset{Z_l}{\lhd} K \underset{m}{<} \Gamma}\phi(l) = \sum_{l|n}\sum_{P \underset{n}{<} \Gamma}\sum_{P \underset{Z_l}{\lhd} K \underset{m}{<} \Gamma}\phi(l)$$

$$= \sum_{l|n}\sum_{K \underset{m}{<} \Gamma}\sum_{P \underset{Z_l}{\lhd} K}\phi(l) = \sum_{l|n}\sum_{K \underset{m}{<} \Gamma}|\mathrm{Epi}(K, Z_l)|.$$

The last equality is implied by the following observation. If $P \underset{Z_l}{\lhd} K$, then there exist precisely $\phi(l)$ epimorphisms $\psi: K \to Z_l$ with $\mathrm{Ker}(\psi) = P$.

Let $\mathrm{Hom}(\Gamma, Z_l)$ be the set of all epimorphisms from the group Γ onto the cyclic group Z_l of order l. Since $|\mathrm{Hom}(\Gamma, Z_l)| = \sum_{d|l}|\mathrm{Epi}(\Gamma, Z_d)|$, the Möbius inversion formula implies the following result due to Jones [1].

Lemma 3. *The following equality holds*:

$$|\mathrm{Epi}(\Gamma, Z_l)| = \sum_{d|l}\mu\left(\frac{l}{d}\right)|\mathrm{Hom}(\Gamma, Z_d)|,$$

where $\mu(n)$ *is the Möbius function.*

This lemma substantially simplifies the calculation of $|\mathrm{Epi}(\Gamma, Z_l)|$ for a finitely generated group Γ. Indeed, let $H_1(\Gamma) = \Gamma/[\Gamma, \Gamma]$ be the first homology group of Γ. Suppose that $H_1(\Gamma) = Z_{m_1} \oplus Z_{m_2} \oplus \ldots \oplus Z_{m_s} \oplus Z^r$. Then, the following lemma is valid.

Lemma 4. *The following equality holds*:

$$|\mathrm{Epi}(\Gamma, Z_l)| = \sum_{d|l}\mu\left(\frac{l}{d}\right)(m_1, d)(m_2, d)\ldots(m_s, d)d^r,$$

where (m, d) is the greatest common divisor of m and d.

Proof. Note that $|\mathrm{Hom}(Z_m, Z_d)| = (m, d)$ and $|\mathrm{Hom}(Z, Z_d)| = d$. Since the group Z_d is Abelian, we have

$$\begin{aligned}|\mathrm{Hom}(\Gamma, Z_d)| &= |\mathrm{Hom}(H_1(\Gamma), Z_d)|\\ &= (m_1, d)(m_2, d)\dots(m_s, d)d^r.\end{aligned}$$

The required assertion follows from Lemma 3.

In particular, we obtain the following corollary.

Corollary 1. (*i*) *Let F_r be the free group of rank r. Then, $H_1(F_r) = Z^r$ and $|\mathrm{Epi}(F_r, Z_l)| = \sum_{d|l}\mu\left(\frac{l}{d}\right)d^r$.*

(*ii*) *Let $\Phi_g = \left\langle a_1, b_1, a_2, b_2, \dots, a_g, b_g\colon \prod_{i=1}^{g}[a_i, b_i] = 1\right\rangle$ be the fundamental group of a closed orientable surface of genus g. Then, $H_1(\Phi_g) = Z^{2g}$ and $|\mathrm{Epi}(\Phi_g, Z_l)| = \sum_{d|l}\mu\left(\frac{l}{d}\right)d^{2g}$.*

(*iii*) *Let $\Lambda_p = \left\langle a_1, a_2, \dots, a_p : \prod_{i=1}^{p} a_i^2 = 1\right\rangle$ be the fundamental group of a closed nonorientable surface of genus p. Then, $H_1(\Lambda_p) = Z_2 \oplus Z^{p-1}$ and*

$$|\mathrm{Epi}(\Lambda_p, Z_l)| = \sum_{d|l}\mu\left(\frac{l}{d}\right)(2, d)d^{p-1}.$$

COUNTING COVERINGS OVER SURFACES

Recall that the fundamental group $\pi_1(\mathcal{B})$ of a surface with boundary $\mathcal{B}$ of Euler characteristic $\chi = 1 - r$, where $r \geq 0$, is the free group F_r of rank r. An example of such a surface is the disk $\mathcal{D}_r$ with r holes. The first corollary to Theorem 1 is the following result, which was obtained earlier by Liskovets [6] by using complicated combinatorial considerations.

Theorem 2. *Let $\mathcal{B}$ be a surface with boundary with fundamental group $\pi_1(\mathcal{B}) = F_r$.*

Then, the number of nonequivalent n-fold coverings of $\mathcal{B}$ equals

$$N(n) = \frac{1}{n}\sum_{\substack{l|n\\ lm = n}}\sum_{d|l}\mu\left(\frac{l}{d}\right)d^{(r-1)m+1}M(m),$$

where $M(m)$ is the number of subgroups of index m in the group F_r.

Proof. Note that all subgroups of index m in F_r are isomorphic to $\Gamma_m = F_{(r-1)m+1}$. Theorem 1 and Corollary 1 (i) imply

$$N(n) = \frac{1}{n}\sum_{\substack{l|n\\ lm = n}}|\mathrm{Epi}(\Gamma_m, Z_l)|\cdot M(m),$$

where

$$|\mathrm{Epi}(\Gamma_m, Z_l)| = \sum_{d|l}\mu\left(\frac{l}{d}\right)d^{(r-1)m+1},$$

which completes the proof.

Setting $M(1) = 1$, we obtain the number of subgroups of index m in the group F_r by the recursive formula of Hall [3]:

$$M(m) = m(m!)^{r-1} - \sum_{j=1}^{m-1}(m-j)!^{r-1}M(j).$$

The following result was obtained in [10] by a fairly complicated method.

Theorem 3. *Let $\mathcal{S}$ be a closed orientable surface with fundamental group $\pi_1(\mathcal{S}) = \Phi_g$.*

Then, the number of nonequivalent n-fold coverings of $\mathcal{S}$ is

$$N(n) = \frac{1}{n}\sum_{\substack{l|n\\ lm = n}}\sum_{d|l}\mu\left(\frac{l}{d}\right)d^{2(g-1)m+2}M(m),$$

where $M(m)$ is the number of subgroups of index m in the group Φ_g.

Proof. Let us apply the Riemann–Hurwitz formula, according to which all subgroups of index m in Φ_g are isomorphic to the group $K_m = \Phi_{(g-1)m+1}$. By Theorem 1, we have

$$N(n) = \frac{1}{n}\sum_{\substack{l|n\\ lm = n}}|\mathrm{Epi}(K_m, Z_l)|\cdot M(m),$$

where

$$|\mathrm{Epi}(K_m, Z_l)| = \sum_{d|l}\mu\left(\frac{l}{d}\right)d^{2(g-1)m+2}$$

is determined by using Corollary 1 (ii).

Let $\mathcal{N}$ be a closed nonorientable surface of genus p with fundamental group $\pi_1(\mathcal{N}) = \Lambda_p$, and let $\mathcal{N}_m^+$ and $\mathcal{N}_m^-$ be orientable and nonorientable m-fold coverings of $\mathcal{N}$, respectively. We set $\Gamma_m^+ = \pi_1(\mathcal{N}_m^+)$ and $\Gamma_m^- = \pi_1(\mathcal{N}_m^-)$. For convenience, we refer to Γ_m^+ and Γ_m^- as the orientable and nonorientable subgroups of index m in Λ_p, respectively. The Riemann–Hurwitz formula gives $2\gamma(\mathcal{N}_m^+) - 2 = m(p-2)$ and $\gamma(\mathcal{N}_m^-) - 2 = m(p-2)$, where $p = \gamma(\mathcal{N})$ is the genus of the surface $\mathcal{N}$.

Therefore, $\Gamma_m^+ = \Phi_{\frac{m}{2}(p-2)+1}$ and $\Gamma_m^- = \Lambda_{m(p-2)+2}$.

By Theorem 1, the number $N(n)$ of nonequivalent n-fold coverings is

$$N(n) = \frac{1}{n}\sum_{\substack{l|n\\ lm = n}} (\left|\mathrm{Epi}(\Gamma_m^+, Z_l)\right| \cdot M^+(m)$$

$$+ \left|\mathrm{Epi}(\Gamma_m^-, Z_l)\right| \cdot M^-(m)),$$

where $M^+(m)$ and $M^-(m)$ are the numbers of orientable and nonorientable subgroups of index m in the group Λ_p, respectively.

Corollary 1, (ii) and (iii) imply

$$\left|\mathrm{Epi}(\Gamma_m^+, Z_l)\right| = \sum_{d|l}\mu\left(\frac{l}{d}\right)d^{m(p-2)+2},$$

$$\left|\mathrm{Epi}(\Gamma_m^-, Z_l)\right| = \sum_{d|l}\mu\left(\frac{l}{d}\right)(2, d)d^{m(p-2)+1}.$$

Thus, we have proved the following theorem, which was obtained earlier [12] by using a nontrivial combinatorial technique.

Theorem 4. *Let $\mathcal{N}$ be a closed orientable surface with fundamental group $\pi_1(\mathcal{N}) = \Lambda_p$.*

Then, the number of nonequivalent n-fold coverings is determined by

$$N(n) = \frac{1}{n}\sum_{\substack{l|n\\ lm = n}}\sum_{d|l}\mu\left(\frac{l}{d}\right)$$

$$\times (d^{m(p-2)+2}M^+(m) + (2, d)d^{m(p-2)+1}M^-(m)),$$

where $M^+(m)$ and $M^-(m)$ are the numbers of orientable and nonorientable subgroups of index m in Λ_p, respectively.

For completeness, note that if $\Gamma = \Phi_g$ or Λ_p, then the number $M(m) = M_\nu(m)$ of subgroups of index m in the group Γ equals

$$m\sum_{s=1}^{m}\frac{(-1)^{s+1}}{s}\sum_{\substack{i_1 + i_2 + \ldots + i_s = m\\ i_1, i_2, \ldots, i_s \ge 1}}\beta_{i_1}\beta_{i_2}\ldots\beta_{i_s},$$

where $\beta_k = \sum_{\chi \in D_k}\left(\frac{k!}{f^\chi}\right)^\nu$, D_k is the set of irreducible representations of the symmetry group S_k, f^χ is the degree of the representation χ, and $\nu = 2g - 2$ or $\nu = p - 2$, respectively [10, 12]. Moreover, in the latter case, $M^+(m) = 0$ if m is odd, $M^+(m) = M_{2\nu}\left(\frac{m}{2}\right)$ if m is even, and $M^-(m) = M(m) - M^+(m)$. The number of subgroups can also be found by the recursive formula

$$M(m) = m\beta_m - \sum_{j=1}^{m-1}\beta_{m-j}M(j),\ M(1) = 1.$$

NONEQUIVALENT COVERINGS OF MANIFOLDS

In this section, we assume that all manifolds are connected and have finitely generated fundamental groups. We impose no constraints on their dimensions. Manifolds may be closed or open, orientable or nor, and they may have or may not have boundary. The following theorem is a topological version of Theorem 1.

Theorem 5. *Let $\mathcal{M}$ be a connected manifold with finitely generated fundamental group $\Gamma = \pi_1(\mathcal{M})$.*

Then, the number of nonequivalent n-fold coverings $\mathcal{M}$ equals

$$N(n) = \frac{1}{n}\sum_{\substack{l|n\\ lm = n}}\sum_{\Phi \in \mathfrak{F}_m}\left|\mathrm{Epi}(\Phi, Z_l)\right| \cdot M_{\Phi,\Gamma}(m),$$

where $\mathfrak{F}_m$ is the set of fundamental groups of the m-fold coverings of $\mathcal{M}$ and $M_{\Phi,\Gamma}(m)$ is the number of subgroups of index m in Γ isomorphic to the group Φ.

As a corollary to Theorem 5 and Lemma 3, we obtain the following result.

Theorem 6. *Let $\mathcal{M}$ be a connected manifold with finitely generated fundamental group $\Gamma = \pi_1(\mathcal{M})$.*

Then, the number of nonequivalent n-fold coverings of $\mathcal{M}$ equals

$$N(n) = \frac{1}{n}\sum_{\substack{l|n\\ lm = n}}\sum_{\Phi \in \mathfrak{F}_m}\sum_{d|l}\mu\left(\frac{l}{d}\right)$$

$$\times \left|\mathrm{Hom}(\Phi, Z_d)\right| \cdot M_{\Phi,\Gamma}(m),$$

where $\mathfrak{F}_m$ is the set of fundamental groups of the m-fold coverings of $\mathcal{M}$ and $M_{\Phi,\Gamma}(m)$ is the number of subgroups of index m in Γ isomorphic to the group Φ.

Suppose that $\mathcal{M}$ is a manifold, $\Gamma = \pi_1(\mathcal{M})$, and $H_1(\Gamma) = \Gamma/[\Gamma, \Gamma]$ is the first homology group of $\mathcal{M}$. Since the group Z_d is Abelian, there exists a one-to-one correspondence between the sets $\mathrm{Hom}(\Gamma, Z_d)$ and $\mathrm{Hom}(H_1(\Gamma), Z_d)$. This gives the following homological version of Theorem 6.

Theorem 7. *Let $\mathcal{M}$ be a connected manifold with finitely generated fundamental group $\Gamma = \pi_1(\mathcal{M})$.*

Then, the number of nonequivalent n-fold coverings of $\mathcal{M}$ equals

$$N(n) = \frac{1}{n}\sum_{\substack{l|n \\ lm = n}}\sum_{H \in \mathfrak{H}_m}\sum_{d|l}\mu\left(\frac{l}{d}\right)$$
$$\times |\mathrm{Hom}(H, \mathrm{Z}_d)| \cdot M'_{H,\Gamma}(m),$$

where $\mathfrak{H}_m$ is the set of homology groups of the m-fold coverings of $\mathcal{M}$ and $M'_{H,\Gamma}(m)$ is the number of subgroups F of index m in the group Γ with $H_1(F)$ isomorphic to H.

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