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ENUMERATION OF BRANCHED COVERINGS OF CLOSED ORIENTABLE SURFACES WHOSE BRANCH ORDERS COINCIDE WITH MULTIPLICITY

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Abstract

The number $N_{n,g,r}$ of nonisomorphic n-fold branched coverings of a given closed orientable surface S of genus g with $r \geqq 1$ branch points of order n is determined. The result is given in terms of the Euler characteristic of the surface S with r points removed and the von Sterneck–Ramanujan function $\Phi(k,n) = \sum_{(d,n)=1} \exp\left(\frac{2\pi ikd}{n}\right)$. More precisely, if $\nu = 2g-2+r$ then

$$N_{n,g,r} = \sum_{\ell|n,\ \ell m=n} (m!\,\ell^m)^\nu \sum_{k=1}^{\ell} \left(\frac{\Phi(k,\ell)}{n}\right)^r \sum_{s=0}^{m-1} (-1)^{sr} \binom{m-1}{s}^{-\nu}.$$

1. Introduction

Throughout this paper, a surface means a compact connected 2-manifold without boundary.

A continuous map $\pi : T \to S$ from a surface T onto S is called a *branched covering* if there exists a finite subset $B = \{b_1, b_2, \dots, b_r\}$ of S such that the restriction of π on $T - \pi^{-1}(B)$, $\pi|_{T-\pi^{-1}(B)} : T - \pi^{-1}(B) \to S - B$, is a covering projection in the usual sense. The smallest subset B of T which has this property is called the *branch set.* At each point $x \in \pi^{-1}(B)$, the

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projection π is topologically equivalent to the complex map $z \to z^k$ for a natural number k. We call x the *branch point* of π and the number k the *order* of x. Denote by s_k^p the number of branch points of order k of the map π in the preimage $\pi^{-1}(b_p)$, where $p = 1, 2, \dots, r$ and $k = 1, 2, \dots, n$. We call the $r \times n$ matrix $\sigma = (s_k^p)$ the *ramification type* of the covering π. Two branched coverings $\pi : T \to S$ and $\pi' : T' \to S$ are equivalent (or isomorphic) if there exists a homomorphism $h : T \to T'$ such that $\pi = \pi' \circ h$.

Let S and σ be as above and let g be the genus of the surface S. Then, the classical Hurwitz enumeration problem can be stated in the following way.

Hurwitz enumeration problem. *Determine the number $N_{n,g,\sigma}$ of nonequivalent coverings of multiplicity n of a surface S of genus g with a given ramification type σ.*

Hurwitz [7], [8] constructed a generating function for the number of nonequivalent coverings of the sphere having only simple points (of order two) and proved that the number of such coverings can be expressed in terms of irreducible characters of the symmetric group. Röhrl [20] obtained upper and lower estimates for the number of nonequivalent coverings with a given ramification type. Some partial solutions of the problem were obtained in [10]-[15] and [19]. In particular, the number of coverings with a given branch set without restriction on the ramification type were obtained in [12]. The complete solution of the Hurwitz enumeration problem is contained in [16]. The solution is given in terms of irreducible characters of the symmetric group which makes it very complicated. It was known just a few cases [17], [18], [10], [11] when it is possible to avoid characters of symmetric groups for calculation the number of coverings. Recently, some new results (see, for example [5] and [6]) were obtained to make it clear that, in many cases, the number of coverings can be expressed in terms of the number theoretical functions. In the present paper, we will show that this takes a place for the covering whose branch orders coincide with the multiplicity. In two partial cases (see Corollary 1 and Corollary 2 below) this result was obtained early by the second named author [18].

2. Preliminaries

Let $\pi : T \to S$ be a branched covering with branch set $B = \{b_1, b_2, \dots, b_r\}$ of ramification type $\sigma = (s_k^p)_{\substack{p=1,2,\dots,r,\\ k=1,2,\dots,n}}$. The set B will be considered fixed in what follows. We denote by S_n the symmetric groups on n symbols and by $(1^{s_1} 2^{s_2} \cdots n^{s_n})$ a permutation from S_n consisting of s_k cycles of length k, $k = 1, 2, \dots, n$. It follows from the results of Hurwitz that each covering π with ramification type σ is uniquely defined by the ordered tuple of permutations

$$(1)\quad \bigl(a_1, b_1, \ldots, a_g, b_g, (1^{s_1^1} \cdots n^{s_n^1}), (1^{s_1^2} \cdots n^{s_n^2}), \ldots, (1^{s_1^r} \cdots n^{s_n^r})\bigr) \in S_n^{2g+r}$$

satisfying the relation

$$\prod_{i=1}^{g} [a_i, b_i] \prod_{p=1}^{r} (1^{s_1^p} 2^{s_2^p} \cdots n^{s_n^p}) = 1 \tag{2}$$

and generating a transitive subgroup of S_n (transitive tuples). Two coverings are equivalent if and only if the corresponding tuples are conjugate via a permutation from S_n. The proof of these facts can be found, for example, in [3] and [12].

Denote by $\mathcal{B}_{n,g,\sigma}$ the set of all tuples of the form (1) satisfying equation (2) and select in $\mathcal{B}_{n,g,\sigma}$ a subset $\mathcal{T}_{n,g,\sigma}$ formed by transitive tuples. We set $B_{n,g,\sigma} = |\mathcal{B}_{n,g,\sigma}|$ and $T_{n,g,\sigma} = |\mathcal{T}_{n,g,\sigma}|$, where $|X|$ denotes the cardinality of a set X. The following results have been obtained in [16].

THEOREM 1. *The number $B_{n,g,\sigma}$ of elements of the set $\mathcal{B}_{n,g,\sigma}$ is defined by the formula*

$$n! \sum_{\lambda \in D_n} \prod_{p=1}^{r} \frac{\chi^{\lambda}_{s_1^p s_2^p \cdots s_n^p}}{1^{s_1^p} s_1^p! \cdot 2^{s_2^p} s_2^p! \cdots n^{s_n^p} s_n^p!} \left(\frac{n!}{f^{\lambda}}\right)^{2g-2+r}, \tag{3}$$

where D_n is the set of all irreducible representations of the group S_n, f^{λ} is the degree and $\chi^{\lambda}_{s_1^p s_2^p \cdots s_n^p}$ the character of the permutation $(1^{s_1} 2^{s_2} \cdots n^{s_n})$ corresponding to the representation λ.

THEOREM 2. *The number $T_{n,g,\sigma}$ of elements of the set $\mathcal{T}_{n,g,\sigma}$ is defined by the formula*

(4)
$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \sum_{\substack{n_1+n_2+\ldots+n_k=n \\ \sigma_1+\sigma_2+\cdots+\sigma_k=\sigma}} \binom{n}{n_1, n_2, \ldots, n_k} B_{n_1,g,\sigma_1} \cdot B_{n_2,g,\sigma_2} \cdots B_{n_k,g,\sigma_k}.$$

Denote by $\mu(n)$, $\varphi(n)$ and $\Phi(x,n)$ the Möbius, Euler and von Sterneck–Ramanujan functions respectively. The relationship between them is given by the formula

$$\Phi(x,n) = \frac{\varphi(n)}{\varphi\left(\frac{n}{(x,n)}\right)} \mu\left(\frac{n}{(x,n)}\right),$$

where (x,n) is the greatest common divisor of x and n. It was shown by O. Hölder that $\Phi(x,n)$ coincides with the Ramanujan sum $\sum\limits_{(d,n)=1} \exp\left(\frac{2\,ikd}{n}\right)$. For the proof, see ([1], p. 164).

3. Coverings whose orders coincide with multiplicity

In this section, we consider n-fold coverings of a compact oriented surface S of genus g having r branch points of order n. We will assume that $r \geqq 1$. The case $r = 0$ was considered in [19]. The ramification type of the covering under investigation is given by the matrix $\sigma = (s_k^p)_{\substack{p=1,2,\ldots,r\\ k=1,2,\ldots,n}}$, where $s_n^p = 1$, $p = 1,2,\ldots,r$ and $s_k^p = 0$ if $k = 1,2,\ldots,n-1$. In this case, each of r branch points of the covering is determined by a cycle (n^1) of the length n. To indicate this property we will use notation $\mathcal{B}_{n,g,r}$ instead of $\mathcal{B}_{n,g,\sigma}$ and similar notations for $\mathcal{T}_{n,g,\sigma}$, $T_{n,g,\sigma}$ and $N_{n,g,\sigma}$. Since $r \geqq 1$ each tuple of the set $\mathcal{B}_{n,g,r}$ is transitive. Hence, $\mathcal{B}_{n,g,r} = \mathcal{T}_{n,g,r}$ and the number of elements of the set $\mathcal{T}_{n,g,r}$ is defined by Theorem 1. It gives

Lemma 1. *The number $T_{m,g,r}$ of elements of the set $\mathcal{T}_{m,g,r}$ is defined by the formula*

$$T_{m,g,r} = m! \sum_{\lambda \in D_m} \left(\frac{\chi_m^\lambda}{m}\right)^r \left(\frac{m!}{f^\lambda}\right)^{2g-2+r},$$

where D_m is the set of all irreducible representations of the group S_m, f^λ is the degree and χ_m^λ the character of the cycle (m^1) of the length m corresponding to the representation λ.

Recall that the number of covering $N_{n,g,r}$ coincides with the number of orbit of symmetric group S_n acting by conjugation on the set $\mathcal{T}_{n,g,r}$. By applying Burnside's lemma we obtain ([18], formulae (17) and (18))

Lemma 2. *The number $N_{n,g,r}$ of nonisomorphic n-fold branched coverings of a given closed orientable surface S of genus g with r, $r \geqq 1$ branch points of order n is given by the formula*

$$N_{n,g,r} = \frac{1}{n} \sum_{\substack{\ell | n\\ \ell\, m = n}} \frac{\ell^{(2g-2)m}}{(m-1)!} \cdot T_{m,g,r} \cdot F_r(\ell), \tag{5}$$

where

$$F_r(\ell) = \ell \sum_{x=1}^{\ell} \left(\frac{\Phi(x,\ell)}{\ell}\right)^r, \tag{6}$$

and $T_{m,g,r}$ is the same as in Lemma 1.

Note [18] that the function $F_r(\ell)$ is multiplicative with respect to ℓ and has the following properties

$$F_1(\ell) = \delta_{1,\ell}, \quad F_2(\ell) = \varphi(\ell), \quad F_3(\ell) = \varphi(\ell)\prod_{i=1}^{s}\frac{p_i-2}{p_i},$$

$$F_4(\ell) = \varphi(\ell)\prod_{i=1}^{s}\frac{p_i^2-3p_i+3}{p_i^2},$$

where $\delta_{1,\ell}$ is the Kronecker symbol, $\varphi(\ell)$ is the Euler function and the product is taken over all prime divisors of ℓ. The structure of the function $T_{m,g,r}$ is much more complicated. It was shown in [18] that in two particular cases $(g,r)=(0,3)$ and $(g,r)=(1,1)$ this function can be expressed in number theoretical terms. The following lemma makes us sure that this is true for all $g \geqq 0$ and $r \geqq 1$.

Lemma 3. *The number $T_{m,g,r}$ of elements of the set $\mathcal{T}_{m,g,r}$ is defined by the formula*

$$T_{m,g,r} = \frac{(m!)^{\nu+1}}{m^r}\sum_{s=0}^{m-1}(-1)^{sr}\binom{m-1}{s}^{-\nu}, \tag{7}$$

where $\nu = 2g-2+r$.

Proof. By definition, $T_{m,g,r}$ is the number of solutions of the equation $\prod_{i=1}^{g}[a_i,b_i]\prod_{j=1}^{r}e_j = 1$ in the symmetric group S_m, where $e_j,\ j=1,2,\ldots,r$ are $m-$cycles and $a_i,\ b_i,\ i=1,2,\ldots,g$ are arbitrary substitutions. Note, since $r \geqq 1$, the group generated by $a_i,\ b_i,\ i=1,2,\ldots,g,\ e_j,\ j=1,2,\ldots,r$ is transitive in S_m. By Lemma 1 we have

$$T_{m,g,r} = m!\sum_{\lambda\in D_m}\left(\frac{\chi_m^\lambda}{m}\right)^r\left(\frac{m!}{f^\lambda}\right)^{2g-2+r}. \tag{8}$$

For the case $r=0$ this formula was obtained by Frobenius and Schur [4]. To avoid the characters of the symmetric group in our calculations we note ([9], Theorem 21.4) that

$$\chi_m^\lambda = \begin{cases}(-1)^s & \text{if } \lambda\vdash(1^s, m-s),\ 0\leqq s\leqq m-1\\ 0 & \text{otherwise.}\end{cases}$$

By the hook formula [9] for any $\lambda \vdash (1^s, m-s)$, we have

$$f^\lambda = \frac{m!}{s! \cdot m \cdot (m-s-1)!} = \frac{(m-1)!}{s!\,(m-s-1)!} = \binom{m-1}{s}.$$

Putting the last two formulae into (8) we obtain (7). □

We remark that the statement of Lemma 2 for the case $g=0$ can be found in ([21], p. 471).

4. Main results

The main result of this paper is the following

THEOREM 3. *The number $N_{n,g,r}$ of nonisomorphic n-fold branched coverings of a given closed orientable surface S of genus g with r, $r \geqq 1$ branch points of order n is given by the formula*

$$N_{n,g,r} = \sum_{\ell|n,\ \ell m=n} (m!\ell^m)^\nu \sum_{k=1}^{\ell} \left(\frac{\Phi(k,\ell)}{n}\right)^r \sum_{s=0}^{m-1} (-1)^{sr} \binom{m-1}{s}^{-\nu},$$

where $\nu = 2g-2+r$ and $\Phi(k,n) = \sum_{(d,n)=1} \exp\left(\frac{2\pi i k d}{n}\right)$ is the von Sterneck–Ramanujan function.

PROOF. By Lemmas 2 and 3, we have

$$N_{n,g,r} = \frac{1}{n} \sum_{\ell|n,\ \ell m=n} \frac{\ell^{(2g-2+r)m}}{(m-1)!} \cdot T_{m,g,r} \cdot F_r(\ell)$$

$$= \frac{1}{n} \sum_{\ell|n,\ \ell m=n} \frac{\ell^{\nu m}}{(m-1)!} \cdot \frac{(m!)^{\nu+1}}{m^r} \sum_{s=0}^{m-1} (-1)^{sr} \binom{m-1}{s}^{-\nu} \cdot \ell \sum_{x=1}^{\ell} \left(\frac{\Phi(x,\ell)}{\ell}\right)^r$$

$$= \sum_{\ell|n,\ \ell m=n} (m!\,\ell^m)^\nu \sum_{k=1}^{\ell} \left(\frac{\Phi(k,\ell)}{n}\right)^r \sum_{s=0}^{m-1} (-1)^{sr} \binom{m-1}{s}^{-\nu}.$$ □

To obtain some consequences, we need the following elementary lemma.

LEMMA 4. *For $(g,r) = (0,3)$ and $(g,r) = (1,1)$ we have*

$$\sum_{s=0}^{m-1} (-1)^{sr} \binom{m-1}{s}^{-\nu} = \frac{2m}{m+1}\,\delta_{m,\ \mathrm{odd}},$$

where $\delta_{m,\,\mathrm{odd}} = 1$ *if* n *is odd, and is equal to* 0 *otherwise.*

Proof. Note that, in both cases $(g, r) = (0, 3)$ and $(g, r) = (1, 1)$, the number r is odd and $\nu = 2g - 2 + r = 1$. Hence

$$\sum_{s=0}^{m-1} (-1)^{sr} \binom{m-1}{s}^{-\nu} = \sum_{s=0}^{m-1} (-1)^{s} \binom{m-1}{s}^{-1} = \frac{2m}{m+1}\,\delta_{m,\,\mathrm{odd}}.$$

The last formula easily follows from the identity

$$\binom{m-1}{s}^{-1} = \frac{m}{m+1}\left(\binom{m}{s}^{-1} + \binom{m}{s+1}^{-1}\right). \qquad \square$$

As an immediate consequence of the Theorem 3 and Lemma 4, we obtain the following two corollaries [18].

Corollary 1. *The number* $N_{n,0,3}$ *of nonisomorphic* n*-fold branched coverings of the sphere with three branch points of order* n *is given by the formula*

$$N_{n,0,3} = \frac{1}{n} \sum_{\ell | n,\, \ell m = n} \frac{2\ell^m (m-1)!}{m+1}\, F_3(\ell)$$

if n *is odd, and is equal to zero, if* n *is even. We set* $F_3(\ell) = \varphi(\ell) \prod\limits_{i=1}^{s} \frac{p_i - 2}{p_i}$, *where the product is taken over all prime divisors of* ℓ.

Proof. We note by (6) that $\sum_{k=1}^{\ell} \big(\Phi(k,\ell)\big)^3 = \ell^2\, F_3(\ell)$ and $F_3(\ell) = 0$ for ℓ even. Hence $F_3(\ell) = F_3(\ell) \cdot \delta_{\ell,odd}$. By Theorem 3 and Lemma 4 we have

$$\begin{aligned} N_{n,0,3} &= \sum_{\ell | n,\, \ell m = n} m!\, \ell^m \cdot \frac{\ell^2\, F_3(\ell)}{n^3}\, \delta_{\ell,\,\mathrm{odd}} \cdot \frac{2m}{m+1}\, \delta_{m,\,\mathrm{odd}} \\ &= \frac{1}{n} \sum_{\ell | n,\, \ell m = n} \frac{2\ell^m (m-1)!}{m+1}\, F_3(\ell) \cdot \delta_{n,\,\mathrm{odd}}, \end{aligned}$$

which is equivalent to the statement of the corollary. $\square$

Corollary 2. *The number* $N_{n,1,1}$ *of nonisomorphic* n*-fold branched coverings of the torus with one branch point of order* n *is given by the formula*

$$N_{n,1,1} = \frac{2n!}{n+1}$$

if n *is odd, and is equal to zero if* n *is even.*

PROOF. By the formula (6) we have $\sum_{k=1}^{\ell} \Phi(k,\ell) = F_1(\ell)$, where $F_1(\ell) = \delta_{1,\ell}$. Hence, by Theorem 3 and Lemma 4, we obtain

$$N_{n,1,1} = \sum_{\ell|n,\ \ell m=n} m!\,\ell^m \cdot \frac{\delta_{1,\ell}}{n} \cdot \frac{2m}{m+1}\,\delta_{m,\text{ odd}} = \frac{2n!}{n+1} \cdot \delta_{n,\text{ odd}}. \qquad \square$$

Here are the values of $N_{n,g,r}$ for small n:

$$N_{1,g,r} = 1;$$

$$N_{2,g,r} = 2^{2g} \cdot \delta_{r,\text{ even}};$$

$$N_{3,g,r} = 3^{3g-2}\big(2^{\nu+1} + 2^r + (-1)^r \cdot 3\big);$$

$$N_{4,g,r} = 2 \cdot 4^{2g-2}(6^\nu + 3 \cdot 2^\nu + 2^r) \cdot \delta_{r,\text{ even}};$$

$$N_{5,g,r} = 5^{2g-2}\big(2 \cdot 24^\nu + 2(-1)^r \cdot 6^\nu + 4^\nu + 4^r + (-1)^r \cdot 4\big);$$

$$N_{6,g,r} = 2 \cdot 6^{2g-2}(120^\nu + 24^\nu + 12^\nu + 2 \cdot 8^\nu + 4^\nu + 2 \cdot 3^\nu + 3^\nu \cdot 2^r + 2^r + 2) \cdot \delta_{r,\text{ even}};$$

$$N_{7,g,r} = 7^{2g-2}\big(2 \cdot 720^\nu + 2 \cdot (-1)^r \cdot 120^\nu + 2 \cdot 48^\nu + (-1)^r \cdot 36^\nu + 6^r + (-1)^r \cdot 6\big);$$

$$N_{8,g,r} = 2 \cdot 8^{2g-2}(5040^\nu + 720^\nu + 240^\nu + 144^\nu + 2 \cdot 48^\nu + 2 \cdot 16^\nu + 4^\nu \cdot 2^{r+1} + 4^r)\delta_{r,\text{ even}},$$

where $\delta_{r,\text{ even}} = 1$ if r is even, and is equal to 0 otherwise.

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